MA 635, Fall 2015

Assignment 7.

This homework is due *Thursday*, October 22.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper and *credit your collaborators*. Your solutions should contain full proofs. Bare answers will not earn you much. Extra problems (if there are any) are due December 11.

1. Quick reminder

Function $f : E \to \mathbb{R} \cup \pm\infty$ is measurable if E is measurable and preimage of every open set is measurable and preimages of $+\infty$, $-\infty$ are measurable. If E is measurable, the function f is measurable if and only if preimage of every interval of the form $(c, \infty]$ is measurable.

Sum and product of a.e. finite measurable function is measurable. Composition continuous $\circ$ measurable is measurable.

2. Exercises

(1) (3.1.1+) Suppose f and g are continuous on $[a, b]$. Show that if $f = g$ a.e. on $[a, b]$, then, in fact, $f = g$ on $[a, b]$. Is a similar assertion true if $[a, b]$ is replaced by a general measurable set E? By a general measurable set E of nonzero measure?

(2) (3.1.3) Suppose a function f has a measurable domain and is continuous except at a finite number of points. Is f necessarily measurable? (*Hint:* Use in-class theorem about a measurable function on D and $E \setminus D$.)

(3) ($\sim$3.1.4) It was proved in class that for a measurable function f and any $c \in \mathbb{R}$, the set $f^{-1}(c)$ is measurable. Prove that converse is not true: a function f with measurable domain such that $f^{-1}(c)$ is measurable for each $c \in \mathbb{R}$ may be not measurable. (*Hint:* One good way to ensure $f^{-1}(c)$ is measurable is to have a monotone function.)

(4) (3.1.5) Suppose the function f is defined on a measurable set E and has the property that $\{x \in E \mid f(x) > c\}$ is measurable for each rational c. Is f necessarily measurable?

(5) (3.1.7) Let $E \subseteq R$ be measurable and let $f : E \to \mathbb{R}$. Prove that f is measurable if and only if for each Borel set A, $f^{-1}(A)$ is measurable. (*Hint:* Show that the collection of sets B with the property that $f^{-1}(B)$ is measurable is a σ-algebra.)

(6) (3.1.9) Let $\{f_n\}$ be a sequence of measurable functions defined on a set E. Define E_0 to be the set of points of E at which $\{f_n\}$ converges. Is the set E_0 measurable? (*Hint:* $\{f_n\}$ converges if and only if it is Cauchy.)

(7) (3.2.12) Let f be a bounded measurable function on E. Show that there are sequences of simple functions on E, $\{\varphi_n\}$ and $\{\psi_n\}$, such that $\{\varphi_n\}$ is increasing, $\{\psi_n\}$ is decreasing and each of these sequences converge to f uniformly on E. (*Hint:* Use Simple Approximation Lemma.)